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A Convergence Exponent for Multidimensional Continued-Fraction Algorithms

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We study a convergence exponent α of multidimensional continued-fraction algorithms (MCFAs). We provide a dynamical systems interpretation for this exponent, then express a general relation for the exponent in terms of the Kolmogorov–Sinai (KS) entropy and smallest eigenvalue of the associated shift map. We consider the case of approximating two irrationals and demonstrate the numerical method for using the smallest eigenvalue and entropy to evaluate α for several MCFAs, including Jacobi–Perron and GMA (generalized mediant algorithm). On very general grounds, the bounds for this exponent (for two irrationals) are $1 \leqslant \alpha \leqslant 3/2 = 1.5$. The upper bound is attained for algorithms with best approximation properties. We find $\alpha_{GMA} = 1.387$ and $\alpha_{JP} = 1.374$, as well as the values for the KS entropy and Oseledec eigenvalues.

KEY WORDS: Continued fractions; entropy; algorithm.

1. INTRODUCTION

In a recent work, we thoroughly investigated a multidimensional continued-fraction scheme.[1] This algorithm, the generalized mediant algorithm (GMA), has a natural geometric interpretation. Moreover, the algorithm possessed the pleasant feature that we could explicitly solve for the invariant measure and also the Kolmogorov–Sinai entropy for the GMA no matter how many irrationals are to be approximated. This gave, in turn, an analytical form for the growth rate of denominators. In this work, we investigate other metric properties of the GMA as well as metric properties of the Jacobi–Perron algorithm. Both these algorithms share certain features in common: both algorithms give rise to ergodic shift mappings with absolutely continuous invariant measures. Both Euclidean algorithms

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can be built from applications of elementary shear matrices (see Proposition 4). For the present exposition, we consider primarily the case of simultaneously approximating two irrationals and limit ourselves to the question of a well-known convergence exponent. We demonstrate that the convergence exponent α yields important information on the quality of approximations.

In Section 2, we define the exponent α and give a dynamical systems interpretation for it. Known elementary mathematical results on α are briefly reviewed. We also state a theorem that algorithms with common subsequences share the same α. Thus, α is independent of the parametrization of the converging sequence.

In Section 3, we define the shift map associated with this approximation sequence. In order for a sequence of convergents to attain the maximal (optimal) α, there must be a relation among the eigenvalues of this same shift. We state this condition: all eigenvalues of the shift be identical. We also review the algorithms for which we calculate α—the Jacobi–Perron algorithm (and its ordered version) and the GMA.

In Section 4, we present our numerical method for calculating α for the approximation of two irrationals. The method is based on a familiar algorithm for calculating Lyapunov exponents in dynamical systems theory (such as billiards).[10,11] We present numerical results for α for both the GMA and the Jacobi–Perron algorithms for the approximation of two irrationals. There are several significant observations. Foremost, we note that this exponent lies between 1 and 3/2. Since the optimal sequence of convergents to a generic pair of irrationals is known to have an exponent 3/2, then one should infer from the numerical simulations that both the GMA and JP algorithms list nonoptimal convergents (see also ref. 2). In some sense, we should say that the GMA is a better algorithm, since its value for α lies closer to the optimal value of 3/2. Section 5 is a discussion and conclusion. The Appendix provides a proof for Proposition 4 of Section 3.

2. A DYNAMICAL SYSTEMS INTERPRETATION OF THE CONVERGENCE EXPONENT α

Suppose we are given a $d-1$ ordered vector of increasingly ordered irrational numbers $\mathbf{\Omega} = (\omega^{(1)},..., \omega^{(d-1)})$, $0 < \omega^{(1)} < \cdots < \omega^{(d-1)}$. We may write down a similarly ordered set of d-dimensional integer vectors $\mathbf{P}_n = (A_n,..., Y_n, Z_n)$, which may be used to construct an approximation to the vector of irrational numbers $\mathbf{\Omega}$ by writing

$$\mathbf{\Omega}_n = (A_n/Z_n,..., Y_n/Z_n) \tag{2.1}$$

Suppose there exists a prescription for measuring the distance between the vector $\mathbf{\Omega}_n$ and $\mathbf{\Omega}$:

$$\text{dist}(\mathbf{\Omega}_n, \mathbf{\Omega}) \tag{2.2}$$

For a wide class of distance functions dist, it is known that exist sequences $\mathbf{\Omega}_n$ such that

$$\text{dist}(\mathbf{\Omega}_n, \mathbf{\Omega}) \propto \frac{1}{|P_n|^{\alpha}} \tag{2.3}$$

for some exponent α. Here $|\cdot|$ denotes the usual Euclidean norm. For certain sequences, the value of α attains its maximum of $d/(d-1)$. [We shall state (2.3) more precisely later.] This is a well-known result in number theory.

In this section we would like to give a dynamical systems interpretation to this result. Toward this purpose, it is useful to consider first several examples.

Example 1. Linear Difference Equations. We wish to consider approximation methods resulting from the study of linear difference equations. Consider a d th-order linear difference equation

$$L_{n+d} + a_1 L_{n+d-1} + \cdots + a_{d-1} L_{n+1} - L_n = 0 \tag{2.4}$$

where the coefficients a_i are integers, with certain conditions among them, which we state explicitly later [see (2.8)–(2.11)]. Consider a d-dimensional integer vector $\mathbf{P}_n = (A_n, \ldots, Y_n, Z_n)$, where each integer entry (such as B_n) is to separately obey the linear difference equation (2.4). Consider a $(d-1)$-dimensional vector $\mathbf{\Omega}_n$ that is to be formed as in Eq. (2.1). We are interested in the convergence properties of $\mathbf{\Omega}_n$ to its limit $\mathbf{\Omega}$.

In order to evaluate the expression (2.2), we must first specify a distance function. [The exponent α, which we are about to calculate as in (2.3), will be insensitive to the type of norm we choose, for all the commonly studied norms; only the prefactor changes.] To each $(d-1)$-dimensional vector $\mathbf{A} = (\omega_1, \ldots, \omega_{d-1})$ one may uniquely assign a ray in d-dimensional space pointing in the $(\omega_1, \ldots, \omega_{d-1}, 1)$ direction. Given a pair of vectors $(\mathbf{A}, \mathbf{B})$, one may uniquely define (in d-space) a plane containing each of two rays (with common vertex at the origin). We define the distance between the $(d-1)$-vectors as the sine of the angle between these rays on the two-dimensional plane embedded in the d-dimensional space. Expressed solely in terms of the original $(d-1)$-dimensional vectors, one has

$$\text{dist}_{\theta}(\mathbf{A}, \mathbf{B}) = \left[\frac{(\mathbf{A}-\mathbf{B})^2 + A^2 B^2 - (\mathbf{A}\cdot\mathbf{B})^2}{1 + A^2 + B^2 + A^2 B^2} \right]^{1/2}$$

One finds by explicit calculation

$$\lim_{n\to\infty} |P_n|^{\alpha} \operatorname{dist}_{\theta}(\Omega_n, \Omega) = W_{11}^{-1+\alpha/2}[\det(\mathsf{W}_2)]^2 \tag{2.5}$$

$$\alpha = 1 + \frac{\ln|1/\gamma_2|}{\ln|\gamma_1|} \tag{2.6}$$

Here we have defined a self-adjoint matrix

$$\mathsf{W} = (\mathsf{M}^{-1})^T (\mathsf{H}_0)^T \mathsf{H}_0 \mathsf{M}^{-1} \tag{2.7}$$

W_2 is the upper 2 by 2 block of W, and H_0 is the initial (integer) data of the linear difference equation: $\mathsf{H}_{\mathbf{0}_{1j}} = A_j$, $\mathsf{H}_{\mathbf{0}_{2j}} = B_j$, etc. The matrix M is the familiar Van der Monde matrix of linear algebra: $M_{kl} = \gamma_k^{l-1}$. The $\{\gamma_i\}$ are roots of the characteristic equation

$$\gamma^d - a_{d-1}\gamma^{d-1} - a_{d-2}\gamma^{d-2} - \cdots - 1 = 0 \tag{2.8}$$

and the integers a_i [see (2.4)] are such that the Eq. (2.8) is factored

$$\prod_{i=1}^{d} (\gamma - \gamma_i) = 0 \tag{2.9}$$

with

$$|\gamma_1| > 1 > |\gamma_2| \geqslant \cdots \geqslant |\gamma_d| \tag{2.10}$$

The exponent α given by 2.3 is the same as that given by Hua and Yuan.[3] The exponent is insensitive to the type of distance function studied so long as the distance (metric) function gives rise to the usual topology. Comparing (2.8) and (2.9), one also notes

$$\prod_{i=1}^{d} |\gamma_i| = 1 \tag{2.11}$$

There are three important features to be noted from (2.6). The first is that the exponent α involves only the largest two roots of the difference equation. The second feature is that due to Eqs. (2.10), (2.11), we must have

$$1 < \alpha \leqslant \frac{d}{d-1} \tag{2.12}$$

Also observe that $\alpha = d/(d-1)$ if and only if $|\gamma_2| = \cdots = |\gamma_d| = 1/|\gamma_1|^{1/(d-1)}$. The third feature is that if we take a subsequence of the original list of

convergents, the exponent α remains unchanged. For example, suppose that instead of taking as a list of convergents the sequence $\mathbf{\Omega}_n$, we took every other convergent and listed $\mathbf{\Omega}_{2n}$. The new roots would be simply the squares of the roots of the original equation, and would thus leave (2.6) unchanged.

Can we give a simple geometrical picture to the exponent α? Consider successive integer convergents $\mathbf{P}_n, \mathbf{P}_{n+1},..., \mathbf{P}_{n+d-1}$ and the points corresponding to those vectors in d-dimensional space. Since their slopes (with respect to the Z axis) must limit onto a constant $\mathbf{\Omega}$, then the $\mathbf{P}_n$ become stretched along the line $(\mathbf{\Omega}, 1)$. Consider the d-dimensional object (simplex) formed by the origin and these d vertices (in two dimensions a triangle, in three dimensions a pyramid). Obviously this simplex is being stretched so that it contracts upon the line in the direction $(\mathbf{\Omega}, 1)$. The eigenvalue governing this stretching is γ_1. Since the products of the roots are unity, the stretching is a shear and the volume of the primitive simplex is a constant.

There is also a direction in which the elementary simplex most slowly converges upon the line. This contraction is governed by the eigenvalue γ_2. The quantity α is a measure of the rate of this slowest mode of contraction divided by the rate of stretching. Obviously if the contraction is isotropic (all the conjugate roots γ_i, $i \neq 1$, are equal in norm) then α attains the maximum value of $d/(d-1)$. The following theorem would imply that the true listing of optimal convergents gives an isotropic behavior, and the convergence is fast, since α attains its maximum possible value.

Before we proceed farther, we give a more precise definition of best convergents.

Definition 1. *Best convergents.* Let dist be a distance function for finding distances between $(d-1)$-dimensional vectors (with entries less than 1). We say a set of integers P_n is a best convergent (wrt the function dist) to an ordered vector $\Omega(\omega_1,..., \omega_d)$, $0 < \omega_1 < \cdots < \omega_{d-1} < 1$, if, writing $\Omega_n = (A_n/Z_n,..., Y_n/Z_n)$, then $\text{dist}(\Omega_n, \Omega)$ is smaller than any $\text{dist}(\Omega', \Omega)$, where Ω' is formed from a d-dimensional (ordered) integer vector with Euclidean norm less than $|\mathbf{P}_n|$. Thus, there are no integer points closer to the origin in d-space that yield a smaller value for $\text{dist}(\Omega_n, \Omega)$ than does $\mathbf{P}_n$.

Proposition 1. For almost all irrational vectors Ω, there exists a sequence of best convergents Ω_n (with an associated $\mathbf{P}_n$) such that

$$\lim_{n \to \infty} \frac{-\ln \text{dist}(\mathbf{\Omega}_n, \mathbf{\Omega})}{\ln |P_n|} = \alpha_{\max} = \frac{d}{d-1} \tag{2.13}$$

This is a corollary to a famous theorem of Dirichlet on simultaneous Diophantine approximations.[7] Considering Definition 1 and Example 1, the following becomes clear. Even in the case where a vector of irrationals $\Omega = (\omega_1, ..., \omega_{d-1})$ can be constructed from such a linear difference equation, this construction does not necessarily yield best approximations. So the situation is extremely complicated.

There is one situation in which linear difference equations may yield best approximations to an irrational and that we consider in the following example.

Example 2. Convergence to a Cubic Irrational with Complex Conjugate Roots. Consider a linear difference equation

$$L_{n+3} + aL_{n+2} + bL_{n+1} - L_n = 0; \qquad |a+1| \neq |b-1| \tag{2.14}$$

with a characteristic equation

$$\gamma^3 - b\gamma^2 - a\gamma - 1 = 0 \tag{2.15}$$

such that the roots of this characteristic equation are $\{\gamma_i\}$ with $|\gamma_1| > 1 > |\gamma_2| = |\gamma_3|$. That is, there is one real root and a complex conjugate pair of roots. Obviously from Eq. (2.16), $\gamma_1\gamma_2\gamma_3 = 1$, so that $|\gamma_2| = |\gamma_3| = 1/|\gamma_1|^{1/2}$. Suppose then we wish to construct rational approximants to the vector $\Omega = (1/\gamma_1^2, 1/\gamma_1)$. Using $P_n = (L_n, L_{n+1}, L_{n+2})$ and $\Omega_n = (L_n/L_{n+2}, L_{n+1}/L_{n+2})$, from (2.6) we see that α attains its maximum value:

$$\alpha = 1 + \frac{\ln |1/\gamma_2|}{\ln |\gamma_1|} = \frac{3}{2} \tag{2.16}$$

Thus the approximations $\mathbf{\Omega}_n$ yielded by the simple linear equation may be best ($\alpha = 3/2$ is a necessary, but not sufficient condition).

Out goal in this paper is calculate the α's for different MCFAs. This will characterize immediately the quality of the convergents it produces.

We next state an elementary proposition concerning subsequences of convergents.

Proposition 2 (Reparametrization Invariance). Consider a sequence of rational convergents $\{\mathbf{\Omega}_n\}[a(d-1)\text{-vector}]$ formed from integer d-vectors $\mathbf{P}_n = (A_n, ..., Y_n, Z_n)$, by writing $\mathbf{\Omega}_n = (A_n/Z_n, ..., Y_n/Z_n)$. Suppose the following limit exists:

$$\lim_{n \to \infty} \frac{-\ln \operatorname{dist}(\mathbf{\Omega}_n, \mathbf{\Omega})}{\ln |P_n|} = \alpha \tag{2.17}$$

Then subsequences of $\{\mathbf{\Omega}_n\}$ have the same α.

The proof is simple: let Ω_{i_j} be a subsequence of the sequence $\{\Omega_k\}$, where $i_1, i_2, \ldots$ is a positive, strictly increasing sequence of integers. Then define $\alpha_j \equiv -\ln \operatorname{dist}(\boldsymbol{\Omega}_{i_j}, \boldsymbol{\Omega})/\ln |P_{i_j}|$. Since the limit in (2.17) converges to α, then any subsequence such as α_j also converges to α.

We will demonstrate an example of Proposition 2 numerically in Section 4.

3. ERGODIC CONTINUED FRACTIONS

In order to calculate α for an additive continued-fraction-type map, it is convenient to examine the corresponding shift of that map. For example, for the approximation of single irrationals, it is known that the ordinary continued-fraction algorithm satisfies all the best approximation properties (it lists the good convergents). In that case, $d=2$ and one calculates $\lambda_1\lambda_2 = 1$, $\alpha = 1 + \ln |1/\lambda_2|/\ln |\lambda_1| = 2$, so one necessary condition for best approximation sequences is trivially satisfied.

It is not difficult to relate the eigenvalues $\lambda_1, \ldots, \lambda_d$ of the Euclidean algorithms to the eigenvalues of the shift mappings. This has been discussed in a previous work by the author.[1]

Definition 2. *Associated shift of a Euclidean algorithm.* Suppose the ordered sequence of convergents $\mathbf{P}_n = (A_n, \ldots, Y_n, Z_n)$ with $A_n \leqslant B_n \leqslant \cdots$ is formed by writing

$$\mathsf{P}_n^{\dagger} = \hat{\mathsf{K}}_n \mathsf{P}_0^{\dagger} \tag{3.1}$$

where the matrix $\hat{\mathsf{K}}_n$ is a matrix with determinant one with nonnegative integer entries and having one Oseledec eigenvalue λ_1 with norm greater than unity and all other eigenvalues with norm less than unity. We call the procedure for constructing P_n *the Euclidean algorithm.* Suppose there is a well-defined mapping T defined on the rationals by

$$T\left(\frac{A_n}{Z_n}, \ldots, \frac{Y_n}{Z_n}\right) = \left(\frac{A_{n-1}}{Z_{n-1}}, \ldots, \frac{Y_{n-1}}{Z_{n-1}}\right) \tag{3.2}$$

The mapping T is called the *shift map.*

Remark 1. For the cases we consider, T may be extended to the irrationals in a straightforward way.

Proposition 3 (Oseledec[4] Eigenvalues of the Shift). Let T be an ergodic mapping of a subset X of the ordered simplex, $0 \leqslant x_1 \leqslant \cdots \leqslant x_{d-1} \leqslant 1$ to itself, and admitting an absolutely continuous

invariant measure. If T is the associated shift of a Euclidean algorithm as described in Definition 2, then the eigenvalues of T are given by $\{\sigma_i\}^{i=1,\ldots,d-1}$ with $\sigma_i = \lambda_1/\lambda_{d-i+1}$, where $\lambda_1 > 1 > \lambda_2 \geqslant \cdots \geqslant \lambda_d$ are assumed to be the eigenvalues of the Euclidean algorithm. Thus $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_{d-1} > 1$.

We refer the reader to ref. 1. The heart of this paper is contained in the following proposition.

Proposition 4. Consider a Euclidean algorithm with Oseledec eigenvalues, $\lambda_1 > 1 > \lambda_2 \geqslant \cdots \geqslant \lambda_d$. Suppose it has an associated shift (Definition 2) which is ergodic on its invariant set. Then

$$\lim_{n \to \infty} \frac{-\ln \operatorname{dist}(\boldsymbol{\Omega}_n, \boldsymbol{\Omega})}{\ln |P_n|} = \alpha = 1 + \frac{\ln |1/\lambda_2|}{\ln |\lambda_1|} \tag{3.3}$$

The proof is given in the Appendix. This gives us a way to study the convergence exponent α by studying the associated shift. Why? Because Eq. (3.3) gives an expression for α which can be reexpressed in terms of quantities which we can measure from doing a numerical study of the shift, as we will now proceed to show [see (3.11)]. That is the most important point of our paper.

Note from Proposition 3 that all the σ_i are >1. Thus, due to a theorem of Pesin,[8] it is easy to relate the KS entropy of the shift to the largest eigenvalue of the Euclidean algorithm:

$$h = \sum_{i=1}^{d=1} \ln \sigma_i = d \ln \lambda_1 - \ln \prod_{i=1}^{d} \lambda_i = d \ln \lambda_1 - \ln 1 = d \ln \lambda_1 \tag{3.4}$$

This result is consistent with those of Schweiger.[6]

Proposition 5. The KS entropy of an ergodic shift mapping T is related to the largest eigenvalue λ_1 of the Euclidean algorithm by

$$h = d \ln \lambda_1 \tag{3.5}$$

Example 3. The Ordinary Continued Fraction (OCF). For the OCF, the above calculations are straightforward. The OCF shift map reads

$$T(x) = \frac{1}{x} - \left[\frac{1}{x}\right] \tag{3.6}$$

which supports the invariant measure

$$d\mu(x) = \frac{1}{\ln 2} \frac{1}{1+x}\, dx \tag{3.7}$$

The KS entropy is straightforward to calculate:

$$h = \int d\mu(x) \ln \left| \frac{\partial T(x)}{\partial x} \right| = \frac{\pi^2}{6 \ln 2} \tag{3.8}$$

Using $d = 2$, (3.5) and (3.8) imply

$$\ln \lambda_1 = \frac{\pi^2}{12 \ln 2} \tag{3.9}$$

meaning that the denominators Z_n grow in such a manner that

$$\lim_{n \to \infty} \frac{\ln Z_n}{n} \propto \frac{\pi^2}{12 \ln 2} \tag{3.10}$$

which is a well-known result.

In recent years, there have developped a great many ways to calculate the largest (or smallest) eigenvalues (exponents) of ergodic mappings. We can conveniently express the convergence exponent α in terms of the eigenvalues of the shift. Using Propositions 4 and 5, we are lead to the following result.

Proposition 6. The convergence exponent α may be expressed in terms of the KS entropy and the *smallest* exponent of the shift as

$$\alpha = d \frac{\ln \sigma_{d-1}}{h} \tag{3.11}$$

(recall the ordering of the roots $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_{d-1} > 1$).

Proof. Now

$$\alpha = 1 + \frac{\ln(1/\lambda_2)}{\ln \lambda_1} = \frac{\ln(\lambda_1/\lambda_2)}{\ln \lambda_1} = d \frac{\ln(\lambda_1/\lambda_2)}{\mathrm{d} \ln \lambda_1} = d \left(\frac{\ln \sigma_{d-1}}{h} \right)$$

The last equality follows from Propositions 4 and 5.

Proposition 7. The exponent α obtains its maximum value if and only if all the eigenvalues of the shift are equal (which in turn holds if and only if all the $d-1$ eifenvalues of the Euclidean algorithm, which are less than one, are equal).

Proof. Starting from Proposition 3, we know

$$\prod_{i=1}^{d-1} \sigma_i = \lambda_1^{d-1} / (\lambda_2 \cdots \lambda_d) = \lambda_1^d / (\lambda_1 \cdots \lambda_d) = \lambda_1^d = e^h$$

Thus, if all the σ's are equal, then $\sigma_{d-1} = e^{h/(d-1)}$. Using (3.11), then we recover the maximum value for α, $d/(d-1)$. This is the value which characterizes algorithms with best approximation properties.

The entropy h is not difficult to calculate numerically for an expansive mapping such as the shift (for a system with both positive and negative Lyapunov exponents, it is numerically not easy). The largest (or smallest) Lyapunov exponent is also not difficult to calculate numerically (it is the other exponents that can be numerically quite challenging to evaluate). Thus we have expressed the convergence exponent α in terms of quantities which are not difficult to evaluate numerically, since Proposition 3 ensures that the mapping is expansive.

In Section 4, we proceed to calculate α for the following algorithms.

The GMA Algorithm. This algorithm has been discussed at length by the author in a recent work.[1] The Euclidean algorithm is given, for d integers, by

$$\begin{aligned}(A_{n-1}, B_{n-1},..., Y_{n-1}, Z_{n-1}) & \\ = A_n, Z_n - A_n, B_n,..., Y_n \quad & \text{for} \quad A_n \leqslant Z_n - A_n \qquad (3.12)\\ = Z_n - A_n, A_n, B_n,..., Y_n \quad & \text{for} \quad A_n > Z_n - A_n \qquad (3.13)\end{aligned}$$

The associated shift map on $I = d-1$ irrationals is

$$\begin{aligned}T(a,..., z) &= \left(\frac{a}{z}, \frac{1-a}{z}, \frac{b}{z},..., \frac{x}{z}, \frac{y}{z}\right) \quad \text{for} \quad a \leqslant \tfrac{1}{2} \\ &= \left(\frac{1-\alpha}{z}, \frac{a}{z}, \frac{b}{z},..., \frac{x}{z}, \frac{y}{z}\right) \quad \text{for} \quad \frac{1}{2} < a < 1 \qquad (3.14)\end{aligned}$$

depending on the relative magnitude of a and $1-a$.

This shift map supports an invariant measure on an invariant set $\mathscr{S}$ defined by

$$1 > z \geqslant y \geqslant \cdots \geqslant b \geqslant a \geqslant 1-b \qquad (3.15)$$

The invariant measure is given by

$$d\mu(a, b,..., y, z) = \frac{1}{\text{norm}(I)} \frac{da}{a} \cdots \frac{dz}{z} \qquad (3.16)$$

where the normalization constant may be explicitly calculated as

$$\text{norm}(I) = \int_0^1 \frac{dt}{t} \frac{[\ln(1+t)]^{I-1}}{(I-1)!} \qquad (3.17)$$

The Kolmogorov–Sinai entropy may also be conveniently expressed in terms of I and norm (I)

$$h(I) = (I+1)\frac{\text{norm}(I+1)}{\text{norm}(I)} \tag{3.18}$$

The GCFP Algorithm. We mention an algorithm which lists convergents which are a subsequence of those listed by GMA. The GCFP Euclidean algorithm is such that it uses enough applications of GMA until the lowest entry is changed. For $I = 2 = d-1$, the GCPF shift reads

$$T(x, y) = \left(\frac{1-kx}{y+x-kx}, \frac{x}{y+x-kx}\right), \qquad k = \left[\frac{1}{x}\right] = \left[\frac{y}{x}\right] \tag{3.19}$$

$$= \left(\frac{y+x-kx}{1+x-kx}, \frac{x}{1+x-kx}\right), \qquad k = \left[\frac{1}{x}\right] = 1 + \left[\frac{y}{x}\right] \tag{3.20}$$

An analytical form for the invariant measure and entropy of GCFP are unknown.

The Jacobi–Perron Algorithm. The multidimensional continued-fraction scheme which has been most studied has been the Jacobi–Perron algorithm. The Euclidean algorithm reads

$$(A', \ldots, Z') = (B - k_B A, \ldots, Z - k_Z A, A) \tag{3.21}$$

where

$$k_B = [B/A], \quad \text{etc.} \tag{3.22}$$

The associated shift for two irrationals reads

$$T(x, y) = \left(\frac{y}{x} - \left[\frac{y}{x}\right], \frac{1}{x} - \left[\frac{1}{x}\right]\right) \tag{3.23}$$

An analytical form for the invariant measure is unknown.

The Jacobi–Perron Algorithm (Ordered Version). This reads precisely the same as the JP except that after each iteration the entries are reshuffled to run from smallest to largest,

$$(A', \ldots, Z') = \text{ordered}(B - k_B A, \ldots, Z - k_Z A, A) \tag{3.24}$$

For two irrationals, the associated shift reads

$$T(x, y) = \text{ordered}\left(\frac{y}{x} - \left[\frac{y}{x}\right], \frac{1}{x} - \left[\frac{1}{x}\right]\right) \tag{3.25}$$

Both ordered and unordered versions give ergodic transformations on the unit simplex. What is called conventionally the Jacobi–Perron algorithm by other authors is the unordered version presented in (3.21) above.

In Section 4, we investigate numerical procedures to calculate α via (3.11).

4. NUMERICAL PROCEDURE, RESULTS—GMA, JP, GCFP

In order to calculate (3.11) for different ergodic fractional linear transformations, we need to calculate the entropy and the smallest eigenvalue of the shift σ_{d-1}. In this section we present results for the algorithms (JP, GMA, GCFP) for two irrationals ($I=2$, $d=I+1=3$).

If a shift map has all eigenvalues greater than one (such as the ergodic shifts described above), then the entropy is numerically very easy to calculate (even if the invariant measure is unknown). Let J represent the Jacobian of the map T, and J_+ the Jacobian of the expanding subspace of T; we have

$$h=\int d\mu(x)\ln\det|J_+|=\int d\mu(x)\ln\det|J| \tag{4.1}$$

since the expanding subspace is precisely the same as the entire space (all exponents are positive). Equating time and phase averages via the Birkhoff ergodic theorem, one writes

$$\begin{aligned} h&=\lim_{M\to\infty}\frac{1}{M}\ln\det J(T^n(\mathbf{x})|(\mathbf{x}))\\ &=\lim_{M\to\infty}\frac{1}{M}\sum_{n=1}^{M}\ln\det J(T^n(\mathbf{x})|T^{n-1}(\mathbf{x})) \end{aligned} \tag{4.2}$$

Observe that h is the sum of scalar quantities evaluated at each time step. This means *we can numerically evaluate h without having to diagonalize matrices with large entries.*

For $d=3$ ($I=2$), we can reexpress α via (3.11) and (2.11) in terms of the entropy and the *largest* eigenvalue of the shift map [(3.11) involves the smallest eigenvalue]. A simple calculation shows

$$\alpha=3\left(1-\frac{\ln\sigma_1}{h}\right) \tag{4.3}$$

(Note that this is a simplifying feature of $d=3$.) Since the largest eigenvalue of a mapping is not difficult to evaluate numerically,[5] (4.3) means *we can numerically evaluate* α *without having to diagonalize matrices with large entries.*

We calculate σ_1, the largest eigenvalue of the GMA shift, via a procedure due to Kubo[9] and implemented successfully in many examples.[10,11] We suppose $(s_0, \psi(s_0))$ is the curve designating the locally unstable fiber passing through the point (x_0, y_0) in the two-dimensional phase space of an ergodic shift T. Let $(s_1, \psi(s_1))$ be the image of that point under T. Elementary calculations yield

$$\frac{ds_1}{ds_0} = \frac{\partial s_1}{\partial s_0} + \frac{\partial s_1}{\partial \psi_0}\frac{d\psi_0}{ds_0} \tag{4.4}$$

and

$$\frac{d\psi_0}{ds_0} = \frac{\partial \psi_0/\partial s_{-1}|_{\psi_{-1}} + \partial \psi_0/\partial \psi_{-1}|_{s_{-1}}\, d\psi_{-1}/ds_{-1}}{\partial s_0/\partial s_{-1}|_{\psi_{-1}} + \partial s_0/\partial \psi_{-1}|_{s_{-1}}\, d\psi_{-1}/ds_{-1}} \tag{4.5}$$

In Eq. (4.5), the collision data immediately prior to $(s_0, \psi_0(s_0))$ enter the expression as well.

The largest exponent of the shift $\ln \sigma_1$ is given by

$$\ln \sigma_1 = \int d\mu(x) \ln \frac{ds_1}{ds_0} \tag{4.6}$$

$$= \lim_{M\to\infty} \frac{1}{M} \sum_{n=1}^{M} \ln \frac{ds_n}{ds_{n-1}} \tag{4.7}$$

As a remarkable consequence of the Birkhoff ergodic theorem, we may begin our numerical simulation with an arbitrary value of $d\psi_{-1}/ds_{-1}$, and carry out the prescription for solving for σ_1, according to (4.4)–(4.7).

We still would like to tabulate more clearly our expressions for (4.2), (4.3), and (4.7). For a two-dimensional map, then $(x_n, y_n) = T(x_{n-1}, y_{n-1})$. We use

$$\frac{ds_{n+1}}{ds_n} = a_{n+1} + c_{n+1}\frac{d\psi_n}{ds_n} \tag{4.8}$$

$$\frac{d\psi_n}{ds_n} = \frac{b_n + d_n\, d\psi_{n-1}/ds_{n-1}}{a_n + c_n\, d\psi_{n-1}/ds_{n-1}} \tag{4.9}$$

with

$$a_n = \frac{\partial x_n}{\partial x_{n-1}}; \qquad b_n = \frac{\partial y_n}{\partial x_{n-1}}; \qquad c_n = \frac{\partial x_n}{\partial y_{n-1}}; \qquad d_n = \frac{\partial y_n}{\partial y_{n-1}} \tag{4.10}$$

$$\det J_n = |a_n d_n - b_n c_n| \tag{4.11}$$

$$h = \lim_{M\to\infty} \frac{1}{M} \sum_{n=1}^{M} \ln[\det J(T^n(\mathbf{x}))] \tag{4.12}$$

$$\ln \sigma_1 = \lim_{M\to\infty} \frac{1}{M} \sum_{n=1}^{M} \ln \frac{ds_n}{ds_{n-1}} \tag{4.13}$$

We evaluate the quantities a_n, b_n, c_n, d_n, and J_n for the aforementionned maps before we present our numerical results.

GMA. For the GMA we find

$x < 1/2$	$x > 1/2$
$a_{n+1} = 1/y_n$	$= b_{n+1}$
$b_{n+1} = -1/y_n$	$= a_{n+1}$
$c_{n+1} = -x_n/y_n$	$= d_{n+1}$
$d_{n+1} = -(1-x_n)/y_n$	$= c_{n+1}$

Also,

$$|\det J_n| = \frac{1}{y_n^3} \tag{4.14}$$

Jacobi–Perron (Ordered Version). For JP one finds

$\{y_n/x_n\} < \{1/x_n\}$	$\{1/x_n\rangle < \{y_n/x_n\}$
$a_{n+1} = -y_n/x_n^2$	$= b_{n+1}$
$b_{n+1} = -1/x_n^2$	$= a_{n+1}$
$c_{n+1} = 1/x_n$	$= d_{n+1}$
$d_{n+1} = 0$	$= c_{n+1}$

Also,

$$|\det J_n| = \frac{1}{x_n^3} \tag{4.15}$$

Jacobi–Perron. This is the unordered version of the above algorithm. Its Jacobian elements are given by the left-hand column of 4.15:

$$a_{n+1} = -\frac{y_n}{x_n^2}; \quad b_{n+1} = -\frac{1}{x_n^2}; \quad c_{n+1} = \frac{1}{x_n}; \quad d_{n+1} = 0; \quad |\det J_n| = \frac{1}{x_n^3} \tag{4.16}$$

GCFP. For GCPF one finds for $[1/x] = [y/x]$

$$a_{n+1} = [-1 + k(1-y)]/\text{den}^2$$
$$b_{n+1} = y/\text{den}^2$$
$$c_{n+1} = -(1-kx)/\text{den}^2$$
$$d_{n+1} = -x/\text{den}^2$$

where

$$k = \left[\frac{1}{x}\right], \qquad \text{den} = x + y - kx$$

and also

$$|\det J_n| = \frac{1}{\text{den}^3} \tag{4.17}$$

For $[1/x] = [y/x] + 1$ (the only other possibility, since $x + y > 1$),

$$a_{n+1} = (y-1)(k-1)/\text{den}^2$$
$$b_{n+1} = 1/\text{den}^2$$
$$c_{n+1} = 1/\text{den}$$
$$d_{n+1} = 0$$

where

$$k = \left[\frac{1}{x}\right], \qquad \text{den} = x + 1 - kx$$

and

$$|\det J_n| = \frac{1}{\text{den}^3} \tag{4.18}$$

5. DISCUSSION, CONCLUSION

Using the procedure outlined in Section 4, we have evaluated numerically the largest eigenvalue of the Euclidean algorithm and the convergence exponent α for the following maps—the GMA shift, the Jacobi–Perron shift, and GCFP. These algorithms are defined in detail at the end of Section 3.

For the case of two irrationals, there are three Oseledec eigenvalues of the Euclidean algorithm, $\lambda_1 > 1 > \lambda_2 \geqslant \lambda_3$ with $\lambda_1\lambda_2\lambda_3 = 1$. There are two eigenvalues of the associated shift $\sigma_1 \geqslant \sigma_2$. They are related to the λ's by $\sigma_1 = \lambda_1/\lambda_3$ and $\sigma_2 = \lambda_1/\lambda_2$. In our numerical procedure, we evaluate $\ln \sigma_1$ and the Kolmogorov–Sinai entropy of the associated shift h. This is enough information to determine all the eigenvalues.

Our results are displayed in Table I. Ten million iterations were carried out to complete the table for GMA and JP. For GCFP, one million

Table I. Results of Numerical Simulation of Three Multidimensional Continued-Fraction Algorithms[a]

Algorithm	λ_1	λ_2	λ_3	$\ln \sigma_1$	Separation	Entropy	α
GMA	1.200	0.9318	0.8942	0.2941	0.02059	0.5475	1.387
GCFP	1.868	0.7852	0.6818	1.008	0.07035	1.875	1.387
JP	3.322	0.6386	0.4173	1.953	0.1518	3.602	1.374
JP (ordered)	5.173	0.6312	0.3062	2.827	0.3616	4.931	1.280

[a] Defined in Section 3. The numerical procedure determines $\ln \sigma_1$ and h, the KS entropy. The other quantities are related to these quantities by simple relations, such as Eqs. (5.1)–(5.5). None of the algorithms attain the upper bound of $\alpha = 3/2$, which is a necessary criterion of algorithms having best approximation properties.

iterations were used (since convergence is quite fast). The entropy for GMA calculated here numerically as a time average corroborates nicely with the result found from considering a phase average in ref. 1 and performing a single integral. Thus, we have a useful check for our numerical procedure. In summary, we estimate the uncertainty for the quantities in Table I to be bounded by a few parts per ten thousand.

All the quantities in the table may be expressed as a function of h and $\ln \sigma_1$. These are the quantities which fall out naturally in the numerical simulation. The eigenvalue λ_1 is related to the entropy as

$$\lambda_1 = e^{h/3} \tag{5.1}$$

We have defined "separation" to be

$$\text{separation} = \frac{1}{2} \ln \frac{\sigma_1}{\sigma_2} = \ln \sigma_1 - \frac{1}{2} h \tag{5.2}$$

The quantity "separation" is informative. For an algorithm with best approximation properties, this quantity must vanish. As discussed in Section 4,

$$\alpha = 3 \left(1 - \frac{\ln \sigma_1}{h} \right) \tag{5.3}$$

We can also construct λ_2 and λ_3:

$$\lambda_2 = \exp(\ln \sigma_1 - 2h/3) \tag{5.4}$$

$$\lambda_3 = \exp(-\ln \sigma_1 + h/3) \tag{5.5}$$

Although JP has a much higher λ_1 than GMA, its α is smaller, implying that the convergents it lists are not as good as those of GMA.

Very roughly, we should think that the λ_1 is so large that we are speeding by better convergents. GMA provides a slower, more thorough search. GCFP has the same α as that for GMA, because GCFP lists convergents which are a subsequence of GMA. This verifies the reparametrization invariance of Proposition 2.

We have presented the first calculations of the convergence exponents for some well-known MCFAs. The results presented here reflect the necessity of developing algorithms for approximating numbers with best approximation properties.

PROOF OF PROPOSITION 4

The claim of Proposition 4 is that

$$\lim_{n\to\infty} \frac{-\ln \operatorname{dist}(\boldsymbol{\Omega}, \boldsymbol{\Omega}_L)}{\ln |P_L|} \equiv \alpha = 1 + \frac{\ln(1/\lambda_2)}{\ln \lambda_1} \tag{A.1}$$

where $P^{(L)}$ is a d-dimensional integer vector $P^{(L)} = (A_L, \ldots, Z_L)$ with $A_L < \cdots < Z_L$ and $\boldsymbol{\Omega}_L = (A_L/Z_L, B_L/Z_L, \ldots, Y_L/Z_L)$ is a $(d-1)$-dimensional vector.

How is the vector $\boldsymbol{\Omega}_L$ constructed? Let $(a_0, a_1, \ldots, a_\infty)$ be the symbol sequence for $\boldsymbol{\Omega}$ in terms of the shift map T. That is, for each a_i, the function $(T_{a_i})^{-1}$ is a well-defined mapping of the invariant set into a subset of that set. For example with GMA,[1] any irrational vector could be expressed in terms of a binary sequence (such as 0100). That is, $T(x, y) = T_0(x, y) = (x/y, (1-x)/y)$ if $x \leqslant 1/2$. If $x > 1/2$, $T(x, y) = T_1(x, y) = ((1-x)/y, x/y)$, and we record whether we use T_0 or T_1 in the binary string. The function T_0 maps $0 < 1-y < x \leqslant 1/2$ to the invariant set, $0 < 1-y < x \leqslant y$. The function T_1 maps $1/2 < x < y < 1$ to the same invariant set. Thus $(T_{a_i})^{-1}$, $a_i \in \{0, 1\}$, is a well-defined and continuous mapping.

The convergents $\boldsymbol{\Omega}_L$ then are simply

$$\boldsymbol{\Omega}_L = (T_{a_0})^{-1} (T_{a_1})^{-1} \cdots (T_{a_L})^{-1} \mathbf{p} \tag{A.2}$$

where $\mathbf{p}$ is a point in the invariant set [for GMA and $d=3$, $\mathbf{p} = (1/2, 1/2)$; any point whose inverse images form an ergodic trajectory throughout the invariant set will do]. Equation (A.2) is clear, since

$$\boldsymbol{\Omega} = \lim_{N\to\infty} (T_{a_0})^{-1} \cdots (T_{a_N})^{-1} \mathbf{p} \tag{A.3}$$

that is, Ω and Ω_L share the first L letters of the symbol sequence. Now using (A.1), we proceed to calculate

$$\alpha = \lim_{L\to\infty} \frac{-1}{L} \ln \operatorname{dist}(\Omega, \Omega_L) \Big/ \lim_{M\to\infty} \frac{1}{M} \ln |P_M| \tag{A.4}$$

$$= \frac{1}{\ln \lambda_1} \lim_{L\to\infty} \frac{-1}{L} \ln \operatorname{dist}(\Omega, \Omega_L) \tag{A.5}$$

$$= \frac{1}{\ln \lambda_1} \lim_{L\to\infty} \frac{-1}{L} \ln \operatorname{dist}((T_{a_0})^{-1} \cdots (T_{a_L})^{-1} \mathbf{p}_L, (T_{a_0})^{-1} \cdots (T_{a_L})^{-1} \mathbf{p}) \tag{A.6}$$

where

$$\mathbf{p}_L = T^L \Omega = T_{a_L} \cdots T_{a_0} \Omega \tag{A.7}$$

Equation (A.5) follows from (A.4) because P_M scales with the denominators so that $(1/M) \ln P_M \to \ln \lambda_1$.

Now define $f_L = (T_{a_0})^{-1} \cdots (T_{a_L})^{-1}$. Then

$$\alpha = \frac{1}{\ln \lambda_1} \lim_{L\to\infty} \frac{-1}{L} \ln \operatorname{dist}(f_L(\mathbf{p}_L), f_L(\mathbf{p})) \tag{A.8}$$

To consider the idea of the proof, let us consider the case where $\mathbf{p}$ and f_L have one component, and the distance function is the normal Euclidean distance. Then we have from (A.8)

$$\alpha = \frac{1}{\ln \lambda_1} \lim_{L\to\infty} \frac{-1}{L} \ln |f_L(p_L) - f_L(p)| \tag{A.9}$$

By the intermediate value theorem there is a point p'_L between p and p_L such that

$$f_L(p_L) - f_L(p) = (p_L - p) \frac{\partial f(p'_L)}{\partial p'_L} \tag{A.10}$$

Then we have from (A.9) and (A.8)

$$\alpha = \frac{1}{\ln \lambda_1} \left(\lim_{L\to\infty} \frac{-1}{L} \ln |p_L - p| + \lim_{L\to\infty} \frac{-1}{L} \ln \left| \frac{\partial f_L(p'_L)}{\partial p'_L} \right| \right) \tag{A.11}$$

The first term in parentheses is zero, since p_L and p are not necessarily points in the invariant set that are close to one another. The second term

is the negative of the Oseledec exponent of the mapping T^{-1}, which yields precisely $\ln \sigma_1 = \ln(\lambda_1/\lambda_2)$. So

$$\alpha = \frac{\ln(\lambda_1/\lambda_2)}{\ln \lambda_1} = 1 + \frac{\ln(1/\lambda_2)}{\ln \lambda_1} \tag{A.12}$$

Actually, since we consider here $d=2$, $\lambda_2 = 1/\lambda_1$ and $\alpha = 2$.

The idea in general dimensions is quite the same. Unfortunately, there is no multidimensional version of the intermediate value theorem that we may use, so that our answer falls out so conveniently as it did above. Let us define the equivalence relation $\simeq$ as follows: $A_L \simeq B_L$ means that

$$\lim_{L\to\infty} \frac{A_L - B_L}{A_L} = \lim_{L\to\infty} \frac{A_L - B_L}{B_L} = 0$$

Consider a term

$$\frac{\partial(f_L(p'_L))_i}{\partial(p'_L)_j} \tag{A.13}$$

where all the $(p'_L)_k$ except $k=j$ are held constant. Remember that f_L is a mapping of a $(d-1)$-dimensional vector to a $(d-1)$-dimensional vector and $(f_L)_i$ is the ith component of this vector. Now

$$\ln \frac{\partial(f_L(p'_L))_i}{\partial(p'_L)_j} \simeq \ln \frac{\partial(f_L(p'_L))_{i'}}{\partial(p'_L)_{j'}} \tag{A.14}$$

That is, as L gets large, both terms on left and right become of order L, with the difference being a constant of order unity, regardless of the values of i, i', j, j'. The reason is that every component of f_L is behaving to the initial conditions in essentially the same manner (see below: Kubo). Let us return to (A.8) and consider the usual Euclidean distance function. All common distance functions will have the same α. Considering (A.8) and (A.14) and a similar application of the intermediate value theorem, one staightforwardly derives

$$\alpha = \frac{-1}{\ln \lambda_1} \lim_{L\to\infty} \frac{1}{L} \ln \left| \frac{\partial(f_L(\mathbf{p}))_1}{\partial \mathbf{p}_{L_1}} \right| \tag{A.15}$$

for some p'_L in the invariant set. Now, the limit in (A.15) is simply the Kubo-like expression[9] for the largest exponent (essentially the formula we use in our numerical work to calculate the largest eigenvalue). Since

$$f_L(p'_L) = (T_{a_0})^{-1} \cdots (T_{a_L})^{-1} p'_L \tag{A.16}$$

Then

$$\alpha = \frac{-1}{\ln \lambda_1} \ln(\text{largest eigenvalue of the } T^{-1} \text{ string}) \tag{A.17}$$

The largest eigenvalue of T^{-1} is the inverse of the smallest eigenvalue of T, the latter being λ_1/λ_2 by Proposition 3. Thus,

$$\alpha = -\frac{\ln(\lambda_1/\lambda_2)^{-1}}{\ln \lambda_1} = 1 + \frac{\ln(1/\lambda_2)}{\ln \lambda_1} \tag{A.18}$$

Proposition 4 is proved.

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